THE MINIMUM FATIGUE LIFE THEOREM

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1. PROGRAM NAME: KFAT

THE KAMAN FATIGUE LIFE OF A STRUCTURAL COMPONENT
SUBJECTED TO REPEATED LOADINGS

1.1 HISTORY:

Original name FAT2, written by Bob Mayerjak in the year of 1966, used on the IBM-370 with
132 character output paper. In 1990 for convenience the input/output formats of FAT2 were
changed by Tam Vothanh to fit personal computers (PC) during the K-1200 helicopter program.
The name has also been changed to KFAT for abbreviation of Kaman Fatigue Program.

1.2 REFERENCES:

1. J. J. Schauble and P. F. Maloney, An Approach to Helicopter Structural Reliability
   and Fatigue Life, AHS, Journal of American Helicopter Society, October 1964
3. Kaman Report no. S-246, Final Fatigue Test Life Calculation of the K747 MRB used
   on the AH-1S, 1981

1.3 SUMMARY

This Fortran program calculates the probable fatigue life of a structural component
subjected to repeated loads.

1.4 BACKGROUND

The Kaman fatigue program named FAT2 (Ref. 2) still exists and has been operational in the last
several decades. This program named KFAT (for Kaman Fatigue) calculates the probable fatigue
life in the same manner as the FAT2 program does, except some modifications, mainly the
input/output formats.

The FAT2 uses output formats of 132 characters printed on the green bar papers by high speed
machines. Although these facilities still exist but are outmoded. The 132 character output may be
printed on PC, using landscape formats and reduced font that are sometimes not favorable. The
KFAT has been rewritten using letter paper formats to ease reading.
THE MINIMUM FATIGUE LIFE THEOREM

Also, for simplicity, many variables in FAT2 that have almost never been used are omitted in the program, like secondary alternate stress, higher harmonics, probability level of input loads other than mean, subgroups of damage accumulation within a load pattern, etc. One may always use FAT2 for these complicated details.

Theoretically the calculated life from KFAT does not differ from that given by FAT2.

1.5 MONTE CARLO TECHNIQUE

The Monte Carlo technique (Ref. 1) that makes the main structure of the both programs. Theoretically a large number (10,000) of pairs of standard deviations, one associated to strength, the other associated to load, are randomly selected, each pair corresponds to a probability level at which the life is calculated. A best fit line of the calculated lives versus probability is drawn, then a life at a desired probability is determined. In practice only 40 pairs have been selected, since it is thought that they yield the more severe results than all others. These 40 pairs of standard deviations used for generating the probability levels of the resulting lives are always attached at the end the input to the FAT2 program.

The original pairs of these numbers are not known. But it is known that these pairs of numbers have changed over the time of more than three decades. Note that only 30 pairs of standard deviations first appeared in the year 1964 (Ref. 2). In fact, comparing these pairs of numbers to those used for the life calculation of the AH-1S helicopter parts (Ref. 3) in the year 1981, none of them match. It is not known from where those 40 pairs of standard deviations got into the AH-1S helicopter FAT2 program. The K747 blade or some other parts of this helicopter fatigue program did not even use the Monte Carlo technique at all, but rather a 30% reduced S/N curve combined with load buckets and cycle count method.

In early years of 1990’s for convenience the input/output formats of FAT2 program were changed. The new program named KFAT was used extensively for the K-1200 helicopter design phase. In the KFAT program, the 40 pairs of standard deviations were built in the program rather than part of the input file, and those pairs were made up of 30 pairs from Ref. 2 and 10 pairs amongst 40 pairs used in the years 1980’s.

1.6 CONSERVATISM OF THE PROGRAM

How does the change of these pairs of standard deviations affect the life calculation? It is difficult to draw a conclusion. The calculated life does not only depend on the standard deviations but also on the coefficients of variation. According to the statistic laws a probability level of the two events (namely here strength and load) may be defined by an infinite number of pairs of standard deviations. Let \( m \) and \( n \) be the numbers of standard deviations, the two pairs
(m,n) and (n,m) yield the same combinatorial probability level at which the life is calculated. But one pair may yield an infinite life, the other a static failure, depending on the associated coefficients of variation. Therefore, for a given probability level, one may select any pair of (m,n) and the calculated life can have any value.

However, given two fixed coefficients of variation, one for strength and one for load, and at a given probability level, there is one and only one pair of (m,n) that yield a minimum life. The program MFAT is based on this property. In the FAT2 and same as in the KFAT program, the pairs (m,n) were selected randomly and thought that they are the severe ones. But it is evident that they do not all yield minimum lives. Furthermore, all pairs yield different combinatorial probability levels that are almost higher than .9995 probability at which the Kaman traditional life is calculated using a linear regression. The highest probability level from the pairs used in both programs is .999967 (associated standard deviation is 3.9925), the 20th in rank is .99960 that is still higher than .9995 (associated standard deviation is 3.2905). In the linear regression plot it is assumed that the calculated lowest life corresponds to .9999 probability (based on the first outcome out of 10,000 drawings). The next higher corresponds to .9998 probability (based on the second outcome without replacement out of 10,000 drawings) and so on for 20 calculated lowest lives. Although a higher combinatorial probability does not necessarily yield a lower life, but it part of sampling (Monte Carlo simulation). Therefore, if the linear regression plot starts from the highest probability associated to the 40 pairs of standard deviations, then the resulting life at .9995 probability would be higher (point B compared to point A, Fig. 1).

Suppose the best fit line of life $L$ versus probability (in term of standard deviations $x$) given by the fatigue program is expressed as

$$\log L = a - b \cdot x$$

Then, the ratio of life at point B over life at point A is simply given by

$$L_B = L_A \cdot 10^{(3.9925 - 3.7195) b}$$

It is seen that the translation of the theoretical regression line to the lower probability yields a conservative result of the calculated life.
Fig. 1 Calculated Life versus Probability
1.7 INPUT FORMAT

The input format of this program is described in the following table.

Table 1. Input format of KFAT fatigue program

<table>
<thead>
<tr>
<th>Rec no.</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Title</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Title</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>NP</td>
<td>NPRINT</td>
<td>NC</td>
<td>RED</td>
<td>PROB</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>TS</td>
<td>TN</td>
<td>RS</td>
<td></td>
<td>repeated</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>RA</td>
<td>FTU</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>Subtitle</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>KL</td>
<td>K1</td>
<td>FSA</td>
<td>FC</td>
<td>FSM</td>
<td>T</td>
<td>BRL</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>Load description</td>
<td>SA</td>
<td>C</td>
<td>SM</td>
<td>RL</td>
<td>repeated</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Rec no. 1:**
Title (anything in 80 spaces)

**Rec no. 2:**
Title (anything in 80 spaces)

**Rec no. 3** (unformatted, all variables must be presented):
- NP (number of points of S/N curve)
- NPRINT: output print control
  - 0 for short print
  - 1 or any positive integer for detail print
- NC : a sample damage calculation output specified by a combination number (from 1 to 40, no damage calculation output if NC is negative)
- RED: a reduction factor applied to the S/N curve, if needed, otherwise RED = 1.
- PROB: probability level (in decimal) at which the life is calculated (used only for MFAT program)
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Rec no. 4 (formatted, repeated NP times):
  • 10 spaces: TS (mean strength of the first S/N curve)
  • 10 spaces: TN (cycles to failure on the S/N curve, input as megacycles)
  • 10 spaces: RS (coefficient of variation for the S/N curve)

Rec no. 5 (unformatted, all variables must be presented):
  • RA (load or stress ratio at which all vibratory loads are to be modified)
  • FTU (material strength used for Goodman modification)
    note: this modification is applied not to any particular load but to all loads, if no Goodman modification is needed, set FTU = 0.

Input of Loading Schedule begins

Rec no. 6:
Subtitle (anything in 80 spaces)

Rec no. 7 (formatted):
  • 1 space: KL
    any single integer (example: 1) if the damage factions of all load cases within the load pattern are taken into account for the fatigue calculation zero or blank if the damage of all load cases of this load pattern is ignored
  • 9 spaces: K1
    number of load cases that follow
  • 10 spaces: FSA
    multiplication factor on vibratory stress (or load) SA
  • 10 spaces: FC
    multiplication factor on cycles (megacycles per kilo hour per 1% schedule time), applied to C and T to obtain the applied cycles of a particular load.

Example: considering a blade rotating 270 rpm or 16,200,000 revolutions per 1000 hours(KH), if a component vibrates N cycles per one revolution of the blade, then that component will go through 16,200,000*N/10^6 = 16.2*N megacycles per 1 KH or .162*N megacycles per 1 KH per 1% schedule.

  • 10 spaces: FSM
    multiplication factor on mean stress (or load) SM
  • 10 spaces: T
    percent of time occupied by the load pattern (based on 100% of time for a complete schedule)
  • 10 spaces: BRL
basic coefficient of variation for each load case in the pattern, overridden by RL

Following are load cases within a load pattern

**Rec no. 8** (formatted, repeated K1 times)

- 20 spaces: DESC
description of load
- 10 spaces: SA
magnitude of the first vibratory stress (or load)
- 10 spaces: C
percent of time of operation occupied by this load within the pattern, i.e., percent of time base on a complete schedule is T*C/100
- 10 spaces: SM
magnitude of the mean stress (or load)
- 10 spaces: RL
coefficient of variation for vibratory stress SA

*Input sequence of record no. 6, record no. 7 and multiple records no. 8 is simultaneously repeated to the end of the schedule.*

### 1.8 EXAMPLE:

**INPUT FILE**

**SAMPLE PROBLEM:** INPUT FORMAT FOR BOTH KFAT AND MFAT PROGRAMS

**SAMPLE LIFE CALCULATION, GOODMAN MODIFICATION TO STRESS RATIO = -1**

<table>
<thead>
<tr>
<th>13</th>
<th>1</th>
<th>1</th>
<th>.9995</th>
</tr>
</thead>
<tbody>
<tr>
<td>32.0</td>
<td>1000.</td>
<td>.220</td>
<td></td>
</tr>
<tr>
<td>35.2</td>
<td>100.</td>
<td>.210</td>
<td></td>
</tr>
<tr>
<td>36.8</td>
<td>50.</td>
<td>.201</td>
<td></td>
</tr>
<tr>
<td>39.2</td>
<td>20.</td>
<td>.193</td>
<td></td>
</tr>
<tr>
<td>41.2</td>
<td>10.</td>
<td>.185</td>
<td></td>
</tr>
<tr>
<td>43.6</td>
<td>5.</td>
<td>.170</td>
<td></td>
</tr>
<tr>
<td>47.2</td>
<td>2.</td>
<td>.162</td>
<td></td>
</tr>
<tr>
<td>50.4</td>
<td>1.</td>
<td>.153</td>
<td></td>
</tr>
<tr>
<td>53.6</td>
<td>.5</td>
<td>.141</td>
<td></td>
</tr>
<tr>
<td>58.8</td>
<td>.2</td>
<td>.130</td>
<td></td>
</tr>
<tr>
<td>62.4</td>
<td>.1</td>
<td>.120</td>
<td></td>
</tr>
<tr>
<td>67.2</td>
<td>.05</td>
<td>.110</td>
<td></td>
</tr>
<tr>
<td>77.6</td>
<td>.01</td>
<td>.100</td>
<td></td>
</tr>
<tr>
<td>-1.</td>
<td>100.</td>
<td>FWD FLIGHT PWR ON</td>
<td></td>
</tr>
</tbody>
</table>
### THE MINIMUM FATIGUE LIFE THEOREM

<table>
<thead>
<tr>
<th>% VNE</th>
<th>Climb Speed</th>
<th>Climb Angle</th>
<th>Climb Rate</th>
<th>Fatigue Life</th>
</tr>
</thead>
<tbody>
<tr>
<td>20%</td>
<td>8.507</td>
<td>1.379</td>
<td>13.0</td>
<td>0.0</td>
</tr>
<tr>
<td>40%</td>
<td>8.904</td>
<td>4.138</td>
<td>15.0</td>
<td>0.337</td>
</tr>
<tr>
<td>60%</td>
<td>9.872</td>
<td>30.345</td>
<td>20.0</td>
<td>0.253</td>
</tr>
<tr>
<td>80%</td>
<td>12.312</td>
<td>38.620</td>
<td>25.0</td>
<td>0.203</td>
</tr>
<tr>
<td>MAX OPERATIONAL</td>
<td>14.863</td>
<td>20.690</td>
<td>0.168</td>
<td></td>
</tr>
<tr>
<td>100%</td>
<td>18.046</td>
<td>4.138</td>
<td>0.139</td>
<td></td>
</tr>
<tr>
<td>111%</td>
<td>23.711</td>
<td>0.690</td>
<td>0.127</td>
<td></td>
</tr>
</tbody>
</table>

**CLIMBS**

<table>
<thead>
<tr>
<th>Climb Speed</th>
<th>Climb Angle</th>
<th>Fatigue Life</th>
</tr>
</thead>
<tbody>
<tr>
<td>14.846</td>
<td>100.0</td>
<td>0.262</td>
</tr>
</tbody>
</table>

Note: italic lines are unformatted.
OUTPUT FILE

PROGRAM NAME: KFAT

_FORMAT MODIFIED FAT2 PROGRAM,
REF. KAMAN REPORT NO. SMR-837)

25-JUL-00

SAMPLE PROBLEM: INPUT FORMAT FOR BOTH KFAT AND MFAT PROGRAMS
SAMPLE LIFE CALCULATION, GOODMAN MODIFICATION TO STRESS RATIO = -1

MEAN S-N CURVE TABLE

<table>
<thead>
<tr>
<th>STRENGTH</th>
<th>CYCLES</th>
<th>CV</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 32.000</td>
<td>1000.000000</td>
<td>0.220</td>
</tr>
<tr>
<td>2 35.200</td>
<td>100.000000</td>
<td>0.210</td>
</tr>
<tr>
<td>3 36.800</td>
<td>50.000000</td>
<td>0.201</td>
</tr>
<tr>
<td>4 39.200</td>
<td>20.000000</td>
<td>0.193</td>
</tr>
<tr>
<td>5 41.200</td>
<td>10.000000</td>
<td>0.185</td>
</tr>
<tr>
<td>6 43.600</td>
<td>5.000000</td>
<td>0.170</td>
</tr>
<tr>
<td>7 47.200</td>
<td>2.000000</td>
<td>0.162</td>
</tr>
<tr>
<td>8 50.400</td>
<td>1.000000</td>
<td>0.153</td>
</tr>
<tr>
<td>9 53.600</td>
<td>0.500000</td>
<td>0.141</td>
</tr>
<tr>
<td>10 58.800</td>
<td>0.200000</td>
<td>0.130</td>
</tr>
<tr>
<td>11 62.400</td>
<td>0.100000</td>
<td>0.120</td>
</tr>
<tr>
<td>12 67.200</td>
<td>0.050000</td>
<td>0.110</td>
</tr>
<tr>
<td>13 77.600</td>
<td>0.010000</td>
<td>0.100</td>
</tr>
</tbody>
</table>

SAMPLE PROBLEM: INPUT FORMAT FOR BOTH KFAT AND MFAT PROGRAMS
SAMPLE LIFE CALCULATION, GOODMAN MODIFICATION TO STRESS RATIO = -1

PATTERNS OF LOADING

1) 72.500 FWD FLIGHT PWR ON

<table>
<thead>
<tr>
<th>CONDITION</th>
<th>VIBRATORY</th>
<th>STEADY</th>
<th>TIME</th>
<th>CV</th>
</tr>
</thead>
<tbody>
<tr>
<td>20% VNE</td>
<td>8.507</td>
<td>13.000</td>
<td>1.000</td>
<td>0.353</td>
</tr>
<tr>
<td>40% VNE</td>
<td>8.904</td>
<td>15.000</td>
<td>3.000</td>
<td>0.337</td>
</tr>
<tr>
<td>60% VNE</td>
<td>9.872</td>
<td>20.000</td>
<td>22.000</td>
<td>0.253</td>
</tr>
<tr>
<td>80% VNE</td>
<td>12.312</td>
<td>25.000</td>
<td>28.000</td>
<td>0.203</td>
</tr>
<tr>
<td>MAX OPERATIONAL</td>
<td>14.863</td>
<td>15.000</td>
<td>0.168</td>
<td></td>
</tr>
<tr>
<td>100% VNE</td>
<td>18.046</td>
<td>3.000</td>
<td>0.139</td>
<td></td>
</tr>
<tr>
<td>111% VNE</td>
<td>23.711</td>
<td>0.500</td>
<td>0.127</td>
<td></td>
</tr>
</tbody>
</table>

2) 27.500 CLIMBS

<table>
<thead>
<tr>
<th>CONDITION</th>
<th>VIBRATORY</th>
<th>STEADY</th>
<th>TIME</th>
<th>CV</th>
</tr>
</thead>
<tbody>
<tr>
<td>COMBINED CLIMBS</td>
<td>14.846</td>
<td>27.500</td>
<td>0.262</td>
<td></td>
</tr>
</tbody>
</table>

SAMPLE PROBLEM: INPUT FORMAT FOR BOTH KFAT AND MFAT PROGRAMS
SAMPLE LIFE CALCULATION, GOODMAN MODIFICATION TO STRESS RATIO = -1

EXTENDED PRINTOUT OF MODIFIED LOAD PATTERNS

1) FWD FLIGHT PWR ON

<table>
<thead>
<tr>
<th>PATTERN</th>
<th>EQUIV.SA</th>
<th>MC/KHR. OF FLIGHT</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9.78</td>
<td>0.162803</td>
</tr>
<tr>
<td>2</td>
<td>10.48</td>
<td>0.488528</td>
</tr>
<tr>
<td>3</td>
<td>12.34</td>
<td>3.582500</td>
</tr>
<tr>
<td>4</td>
<td>16.42</td>
<td>4.559438</td>
</tr>
<tr>
<td>5</td>
<td>14.86</td>
<td>2.442641</td>
</tr>
</tbody>
</table>
2) CLIMBS

PATTERN EQUIV.SA MC/KHR. OF FLIGHT
1 14.85 4.478100

SAMPLE PROBLEM: INPUT FORMAT FOR BOTH KFAT AND MFAT PROGRAMS
SAMPLE LIFE CALCULATION, GOODMAN MODIFICATION TO STRESS RATIO = -1

FOLLOWING SAMPLE CALCULATIONS ARE BASED ON
STANDARD DEVIATIONS APPLIED TO STRENGTH: 3.000 TO LOAD: 1.290
RESULTS FOR DIFFERENT DEVIATION COMBINATIONS OF STRENGTH AND LOAD
ARE TABULATED IN THE TABLE AT THE END OF OUTPUT

1) FWD FLIGHT PWR ON

<table>
<thead>
<tr>
<th>COND.</th>
<th>VIB LOAD</th>
<th>MC/KHR</th>
<th>MEGA CYC</th>
<th>DAMAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ADJUSTED</td>
<td>APPLIED</td>
<td>ALLOWABLE</td>
<td>FRACTION</td>
</tr>
<tr>
<td>20% VNE</td>
<td>14.23</td>
<td>0.162803</td>
<td>59.3670</td>
<td>0.0027</td>
</tr>
<tr>
<td>40% VNE</td>
<td>15.03</td>
<td>0.488528</td>
<td>40.1045</td>
<td>0.0122</td>
</tr>
<tr>
<td>60% VNE</td>
<td>16.37</td>
<td>3.582500</td>
<td>21.2147</td>
<td>0.1689</td>
</tr>
<tr>
<td>80% VNE</td>
<td>20.71</td>
<td>4.559438</td>
<td>5.9344</td>
<td>0.7683</td>
</tr>
<tr>
<td>MAX OPERATIONAL</td>
<td>18.08</td>
<td>2.442641</td>
<td>10.8592</td>
<td>0.2249</td>
</tr>
<tr>
<td>100% VNE</td>
<td>21.28</td>
<td>0.488528</td>
<td>5.1131</td>
<td>0.0955</td>
</tr>
<tr>
<td>111% VNE</td>
<td>27.60</td>
<td>0.081461</td>
<td>0.9391</td>
<td>0.0867</td>
</tr>
<tr>
<td>SUBTOTAL=</td>
<td>1.359320</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2) CLIMBS

<table>
<thead>
<tr>
<th>COND.</th>
<th>VIB LOAD</th>
<th>MC/KHR</th>
<th>MEGA CYC</th>
<th>DAMAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ADJUSTED</td>
<td>APPLIED</td>
<td>ALLOWABLE</td>
<td>FRACTION</td>
</tr>
<tr>
<td>COMBINED CLIMBS</td>
<td>19.86</td>
<td>4.478100</td>
<td>7.2904</td>
<td>0.6142</td>
</tr>
<tr>
<td>SUBTOTAL=</td>
<td>0.614244</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

TOTAL DAMAGE FOR THIS COMBINATION = 1.973564 AND 100.0% OF SPECTRUM
LIFE= 507. FOR THIS DAMAGE
## The Minimum Fatigue Life Theorem

**Sample Problem:** Input format for both Kfat and MFat programs

Sample life calculation, Goodman modification to stress ratio = -1

<table>
<thead>
<tr>
<th>Combination</th>
<th>Standard Strength</th>
<th>Deviation Load</th>
<th>Life Hours</th>
</tr>
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<tbody>
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<td>3.0000</td>
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<tr>
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<td>650.</td>
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<tr>
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<td>0.6070</td>
<td>316.</td>
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<tr>
<td>5</td>
<td>2.1830</td>
<td>2.6530</td>
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<tr>
<td>6</td>
<td>2.7480</td>
<td>1.3480</td>
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<tr>
<td>8</td>
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<td>2.0470</td>
<td>1086.</td>
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<tr>
<td>9</td>
<td>3.3360</td>
<td>0.4140</td>
<td>554.</td>
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<tr>
<td>10</td>
<td>2.9880</td>
<td>0.8380</td>
<td>770.</td>
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<tr>
<td>11</td>
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<td>622.</td>
</tr>
<tr>
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<td>1.9140</td>
<td>1502.</td>
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<tr>
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<td>1.1140</td>
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<tr>
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<tr>
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<td>0.0890</td>
<td>1205.</td>
</tr>
<tr>
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<td>2159.</td>
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<tr>
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<td>39</td>
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<tr>
<td>40</td>
<td>3.0520</td>
<td>-0.1070</td>
<td>2059.</td>
</tr>
</tbody>
</table>

For 20 points, log life = 4.8143 -0.6264 Sigma

Life for 0.999500 Prob. = 566. Hours

The life versus probability is represented by a solid curve in Fig. 1
Fig. 1 Example: Calculated Life versus Probability
2. PROGRAM NAME: MFAT

THEOREM OF MINIMUM LIFE OF A MECHANICAL COMPONENT

2.1 HISTORY:

This program was written by Tam Vothanh in early years of 80’s as an approach to solve the 6-nine probabilistic fatigue program proposed by the Army.

2.2 ABSTRACT

The solution for fatigue life of a mechanical component by maximizing the damage fraction method is presented herein this paper. Considering the combinatorial probability of two events, i.e., material strength and applied load, the damage fraction is a function of four parameters $n$, $u$ and $m$, $v$. They are the number of standard deviations and the coefficients of variation of the strength and load, respectively. At given levels of strength or load, the coefficients of variation, $u$ and $v$, are locally constant, using the combinatorial probability law and setting the total differential of the damage fraction with respect to $m$ and $n$ to zero yield an unique maximum damage fraction caused by the considered load. Then the minimum life of the component is determined using the Miner hypothesis.
THE MINIMUM FATIGUE LIFE THEOREM

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2.3 INTRODUCTION

One of the important structural problems is the determination of the fatigue life of a mechanical component. It is known the life of a component depends on several environmental factors whose effects can not be exactly formulated, like atmospheric pollution, humidity, temperature, acoustic, higher harmonics of loading, material defect, etc. Each factor has its own variance and probability to cause failure. Hence, the problem of fatigue has no definitive solution and still remains in a state-of-art. Let us disregard all other factors and consider only two probabilities, one associated with the applied load, the other associated with the strength of the component material. The component has its life associated with the combinatorial probability of the two individual probabilities. However, from the theory of probability, for a given combinatorial probability there is an infinite number of combinations of the two individual probabilities, and different combinations yield different lives. Therefore, to solve for the component life at a given combinatorial probability by calculating its lives at different combinatorial probabilities then making a plot of life versus probability is completely absurd, only because the life at a given combinatorial probability can be anything from zero to infinite depending on the combination of the two individual probabilities. Furthermore, the calculated life versus probability is nonlinear and not always correlative, an attempt to make any line fit will either fail or yields an inaccurate result. Based on the theory of combinatorial probability and the Miner hypothesis of damage fraction, this writing presents a theoretical approach which yields an exact and unique solution of this problem.

2.4 GENERAL FORMULATION

2.4.1 Strength SN curve

Let the stress/strain versus cycles to failure (SN) curve of a component material be defined as:

\[ \log N = a + b \log (r.s - c) \]  \hspace{1cm} (1)

where \( N \) is the number of cycles to failure, \( a, b \) and \( c \) are the best fit parameters usually derived from test data, \( S \) is the mean strength. The \( c \) is customarily known as the mean endurance limit of the component material. The variable \( r \) is the reduction factor of the strength, defined as:

\[ r = \frac{1}{1 - n.u(s)} \]  \hspace{1cm} (2)

where \( n \) is the number of standard deviations, and \( u(s) \) is the coefficient of variation which may be constant or a function of \( s \). Note that for \( n = 0 \), i.e., \( r = 1 \) the eq. (1) describes the mean SN curve; a positive \( n \) means that the strength of the component material is reduced by \( n \) standard
deviations. For many materials the type of eq. (1) is a standard form used by the Military
Handbook of Materials and Elements for Aerospace Structures (MIL-HDBK-5).

2.4.2 Applied load function

The applied load function may be defined as

\[ S = f \cdot s \]  

(3)

where \( s \) is the mean load and the variable \( f \) is the load factor, defined as:

\[ f = 1 + m \cdot v(s) \]  

(4)

where \( m \) is the number of standard deviations and \( v(s) \) is the load coefficient of variation which
may be a constant or a function of \( s \).

2.4.3 Life calculation based on the Miner hypothesis

Let \( N_s \) be the number of applied cycles associated with load \( s \). In eq. (1) replacing \( s \) by \( S \) from
eq. (3) then solving eq. (1) for \( N \) yield the number of cycles at the load level \( S \). Designate it by \( N_a \),
the allowable cycles. From the Miner hypothesis the damage caused by that load is:

\[ \frac{d}{s} = \frac{N_s}{N_a} \]  

(5)

If there are several cyclic loads in a spectrum, the total damage is the sum of individual damages
of all cyclic loads:

\[ D = \sum d_i \]  

(6)

Then the life is:

\[ L = D^{-1} \]  

(7)

This is the life associated with a particular combination of \((m,n)\) standard deviations of load and
strength, respectively, or, in other words, the life at the combinatorial probability defined by \( m \)
and \( n \).

2.4.4 Life calculation detail

Eq. (1) may be rewritten as:

\[ N_a = 10^a (r \cdot s - c)^b \]  

(8)

Substituting \( s \) by \( S \) from eq. (3) yields the allowable number of cycles at the load level \( S \):

\[ N_a = 10^a (r \cdot f \cdot s - c)^b \]  

(9)

The damage due to load \( i \) having load level \( s_i \) is:

\[ \frac{d_i}{s_i} = \frac{N_i}{N_a} = \frac{N_i}{10^a (s_i \cdot r \cdot f - c)^b} \]  

(10)

Total damage of all loads:
THE MINIMUM FATIGUE LIFE THEOREM

\[ D = \sum \frac{N_u}{10^s (s_y r_f - c)^b} \]  \hspace{1cm} (11)

Then the life is given by eq. (7).

2.5 D FUNCTION

The total damage \( D \) given by eq. (11) is the sum of a number of similar terms. Each of these terms corresponds to a load level deviated from the mean by \( m \) standard deviations defined by the function \( f \), and the strength deviated from the mean by \( n \) standard deviations defined by the function \( r \). Although both coefficients of variation \( u \) and \( v \) are functions of strength and load and they may vary from one term to the other, but for each term they are constant as the corresponding load or strength have a fixed value in that term. Considering the numbers of standard deviations \( m \) and \( n \) as variable parameters, then the value of each term is a function of \( m \) and \( n \). The damage fraction \( D \), or the life, will have an extreme value when the total differential of \( D \) equals to zero, or \( m \) and \( n \) must satisfy the following equation:

\[ \frac{\partial D}{\partial m} + \frac{\partial D}{\partial n} = 0 \]  \hspace{1cm} (12)

One more equation is needed to solve for \( m \) and \( n \).

2.5.1 Combinatorial probability

The combinatorial probability of load and strength is defined according to the probability law as:

\[ P(m,n) = P(m) + P(n) - P(m).P(n) \]  \hspace{1cm} (13)

For a given \( P(m,n) \) the values of \( m \) and \( n \) are not independent, but rather one is a function of the other. The eq. (13) is positive, symmetric, continuous and differentiable with respect to its variables. Considering \( P(m,n) \) as a constant, differentiating \( n \) with respect to \( m \) yields:

\[ \frac{dn}{dm} = \frac{dn}{dP(n)} \frac{dP(n)}{dP(m)} \frac{dP(m)}{dm} \]  \hspace{1cm} (14)

From statistical definition:

\[ P(m) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{2}} dx \]  \hspace{1cm} (15)

Differentiating eq. (15) with respect to \( m \) yields

\[ \frac{dP(m)}{dm} = \frac{1}{\sqrt{2\pi}} e^{-s^2} \]  \hspace{1cm} (16)

and similarly for \( P(n) \):
THE MINIMUM FATIGUE LIFE THEOREM

\[ \frac{dP(n)}{dn} = \frac{1}{\sqrt{2\pi}} e^{-s_n^2} \]  
\[(17)\]

Let \( P(m,n) = P \) be a constant, differentiating eq. (13) yields

\[ \frac{dP(n)}{dP(m)} = \frac{1-P}{[1-P(m)]^2} \]  
\[(18)\]

Substituting eqs. (16), (17), (18) into eq. (14) yields

\[ \frac{dn}{dm} = \frac{1-P}{[1-P(m)]^2} e^{s(n^2-m^2)} \]  
\[(19)\]

This is the equation of the slope of the curve \( n \) as function of \( m \) in the \((m,n)\) plane. The curve \( n \) versus \( m \) can be plotted from eq. (13) by letting \( P(m,n) = P \), a constant, as shown in Fig. 1. Note that the slope of this curve defined by eq. (19) is a monotonic function. Therefore for a given value of \( m \) there is only one corresponding value \( n \), and vice versa, and a pair \((m,n)\) yields an unique value of the slope. This property may be observed from the \((m,n)\) curve in Fig. 1. Also, \( m \) and \( n \) are interchangeable.

2.5.2 Derivative of the D function

The derivative of the D function eq. (11) is the sum of derivatives of the terms in the function. Note that at a given load \( s \) of each term, the values \( u \) and \( v \), functions of \( s \), are locally constant. Therefore each term depends only on the parameters \( m \) and \( n \) which are contained in the functions \( r \) and \( f \). Let us differentiate a typical term of the D function using eq. (12) in which \( D \) is replaced by \( d_i \) from eq. (10):

\[ d(d_i) = \frac{\partial}{\partial m} \left[ \frac{N_u}{10^s(s_r f - c)} \right] dm + \frac{\partial}{\partial n} \left[ \frac{N_u}{10^s(s_r f - c)} \right] dn = 0 \]

Performing this differentiation and simplifying by the term \( \frac{-bN_u s_i}{10^s(s_r f - c)^{b+1}} \) which is always different to zero yield:

\[ r \frac{\partial f}{\partial m} + f \frac{\partial r}{\partial n} = 0 \]

From eqs. (2) and (4):

\[ \frac{\partial r}{\partial n} = \left(1 - nu\right)^2 ; \quad \frac{\partial f}{\partial m} = v \]

Substituting these into the above equation and simplifying yield:

\[ (1-nu) v \, dm + (1+mv) u \, dn = 0 \]  
\[(20)\]

Considering \( u \) and \( v \) as constant at particular load and strength levels, the damage \( d_i \) of the D function is extremized when \( m \) and \( n \) satisfy the eq. (20).
2.5.3 Uniqueness of the solution

The expression on the left hand side of eq. (20) is, by a factor, none other than the total derivative of \( d_i \), i.e.:

\[
d (d_i) = K [(1-\nu) v \, dm + (1+m v) u \, dn]
\]  
(21)

where \( K \) is a factor which has been left out for simplification during the differentiating process:

\[
K = \frac{-bN_{\infty} s_i}{10^{\alpha} (s, r, f - c)^{b+1} (1-\nu)^2}
\]  
(22)

Note that as a property of SN curves \( b \) is a constant and always negative, hence \( K \) is always positive and never equals to zero. Considering \( m \) as an independent variable, the derivative of \( d_i \) with respect to \( m \) is obtained by simply dividing both sides of eq. (21) by \( dm \):

\[
\frac{d(d_i)}{dm} = K[(1-\nu)v + (1+m v) u] \frac{dn}{dm}
\]  
(23)

The eq. (23) is the slope of \( d_i \) considering \( m \) as an independent variable. Since \( K \) never equals zero, the eq. (23) equals zero only when the expression inside the brackets equals zero. The condition eq. (23) is the same as the condition eq. (20) since either \( m \) or \( n \) is the only one independent variable. Designate the expression inside the brackets by \( g \):

\[
g = (1-\nu)v + (1+m v) u \frac{dn}{dm}
\]  
(24)

Differentiating \( g \) with respect to \( m \) yields

\[
\frac{dg}{dm} = (1+m v) u \frac{d^2 n}{dm^2}
\]  
(25)

In eq. (25) above the second derivative of \( n \) with respect to \( m \) (on the right hand side) may be calculated from eq. (19), or more easily, this is simply the curvature of the curve \( n \) as a function of \( m \) in Fig. 1. As a property of the probability equation, it is seen that this curvature never vanishes or changes the sign for all values of \( m \). Therefore \( \frac{dg}{dm} \) never equals zero and hence \( g \) is a monotonic function of \( m \). Since \( K \) is always positive the eq. (23) equals zero only once at one unique value of \( m \). At that value \( m \) (or \( n \), since \( m \) and \( n \) are interchangeable) corresponding to zero derivative of \( d_i \), the value \( d_i \) is extremized.

2.5.4 Maximum D value

It has been proven that the solution is unique and \( d_i \) is extremized only once for all values of \( m \), in other words, for all pairs of \((m,n)\) there is only one pair that make \( d_i \) extremized. The remaining question is whether this extremized value is a maximum or a minimum. To know this, one may look up the curvature of the function \( d_i \) by differentiating eq. (23) with respect to \( m \). Since the \( K \) factor, eq. (22), also contains \( m \) an \( n \) parameters, and \( n \) is a function of \( m \), the second derivative of \( d_i \) with respect to \( m \) is rather complicated. However, let us observe the expression
inside the brackets of eq. (23), expression named \( g \) as defined by eq. (24). When \( m \) becomes large, the second term which contains \( dn/dm \) is predominant and more and more negative. Since \( K \) is always positive and the expression \( g \) is monotonic, the slope of \( d_i \) is negative for large values of \( m \), and it must be positive for small values of \( m \) when \( dn/dm \) tends to zero from the negative side. Therefore the extremized value of \( d_i \) is a maximum. Note that each individual \( d_i \) may be maximized at different pair of \( (m,n) \), although all pairs of \( (m,n) \) correspond to the same given combinatorial probability. It is concluded that the total damage \( D = \Sigma d_i \) has an unique maximum value.

### 2.6 THEOREM OF MINIMUM FATIGUE LIFE

For a continuous and differentiable SN curve, the fatigue life of a mechanical component calculated using the Miner hypothesis and the combinatorial probability law has a minimum life, and this minimum is unique.

Once the SN curve is defined by a continuous and differentiable function, this theorem provides a theoretical exact and unique solution. However, if the SN curve is defined by data points, the minimization of the result is sometimes hesitated, i.e., the result may hit the minimum and rebounds and hits another minimum again. That is because the data curve is not theoretically differentiable or it may not be smooth.

### 2.7 APPLICATION

Given a combinatorial probability level and a material strength with its associated coefficient of variation, it is clear that there is one pair of \( (m,n) \) and only one, such that a specified load with its associated coefficient of variation produces a maximum damage. For another load, there will be another pair of \( (m,n) \) which also an unique one for this load to produce a maximum damage, and so on. From eq. (20) which is the condition of a maximum damage, on may have:

\[
\frac{dn}{dm} = -\frac{(1-nu)v}{(1+nv)u} \tag{26}
\]

Substituting \( dn/dm \) from eq. (26) into eq. (19) yields:

\[
\frac{(1-nu)v}{(1+nv)u} \left( 1 - P \right) e^{\frac{1}{2} (n^2 - m^2)} = 0 \tag{27}
\]

where \( m \) and \( n \) must satisfy eq. (13). Note that for any fatigue problem \( P(m,n) \) in eq. (13) is a given constant, for example \( P(m,n) = P = .9995 \) as depicted in Fig. 1. Knowing \( u \) and \( v \), coefficients of variation of any particular strength and load, the computer will search along the curve eq. (13) for the unique pair of \( (m,n) \) to satisfy the eq. (27), and thus the maximum \( d_i \) is determined.
2.8 CONCLUSION

In this analysis it is presumed that the parameters \( m \) and \( n \) correspond to an increase in load and a reduction of strength, respectively. However, in searching for a maximum damage it may end up to a negative value of \( m \) or \( n \). This means all data on the upper side and lower side of load and strength are used for solution. Theoretically the result is minimum and unique. For a given probability if any other method gives a lower result it is perhaps due to an inaccurate interpolation.

2.9 COMPUTER PROGRAM

2.9.1 Input format

The input format used for this program is exactly the same as that used for KFAT
Fig. 1 Constant Combinatorial Probability Curve of Two Deviations