ABSTRACT

The 1.5 plastic shape factor for a rectangular section, or 1.7 for a circular section, is simple and still on the conservative side of safety. If a component cannot make it, then beef it up. Why have to work so hard? The assumed rectangular plastic stress distribution profile is not absurd but very real, because the slope of the stress strain curve of the elastic part \( \tan^{-1}(E) \) is simply of a squared angle with an error far beyond any engineering tolerance (Figure 1). Theoretically the plastic shape factor is applied \textit{not} the ultimate stress, but to the yield, simply because it is where the material starts to change from elastic to plastic state.

People, who do \textit{practical analysis}, make a complicated thing easy; if one likes to make an easy thing complicated, that one must be a scientist, but… a bad scientist!

Figure 1: An assumed stress profile across a section in full plastic
1. INTRODUCTION

When a material reaches the yield state, at the defined .002 in/in hysteresis strain, if the load is not removed the material undergoes the plastic state. While the stress still increases but not much, the strain keeps increasing probably ten or twenty more times before it fails. The strain at which the material fails is called the *rapture strain*. It is customarily assumed that the deformation is linear with respect to load, and consequently so is the strain. But in the plastic domain, neither stress nor strain is linear with respect to load. For each published stress strain curve from the yield point and beyond, it is not known what the real stress is, when a bar is laterally contracted under a very high axial test force. Using the stress to define the material failure is somewhat vague, but a material definitely fails when it reaches the rapture strain. Figure 1 shows an example of a complete stress strain curve, where it is seen that the material survives a long way over the maximum stress. The mechanical analysis customarily evaluates a component strength based on stresses probably for convenience, but beyond the elastic range, stresses do not mean much.

![Figure 2: 7075-T6 AL Extrusion](image)

The yield is a transient state at which the material starts undergoing the plastic deformation. For the space and/or weight constraints or in the case of rework or retrofit of a part that cannot be changed, the energy of the plastic deformation provides some additional strength that may be needed. This is the becoming of the plastic analysis.

2. PLASTIC BENDING MOMENT

The material plasticity is treated in tons of literatures, text books and technical reports. The customary plastic bending moment factors have been derived from a simple theory as well as confirmed by experimental investigations since two centuries. They have been good enough to be used for all engineering disciplines. If one wants to be involved in the material plasticity, then do something very new.
or more challenging (like plastic flows, cross section distortion, anticlastic curvature due to the Poisson effect, compressible or incompressible materials, non-linear strains flows, loci of max strains, slip-line field, to name a few, but even these are also not new), rather than just calculating the plastic bending moment.

Here only presents a brief understanding for the practical purpose. This is not a theoretical analysis for scientists but only a practical analysis for a designer. The effect of the combined loads, like the axial and/or shear loads in a beam, is disregarded in this writing. Prior to deal with the plastic bending moment, one may look back some work of other authors in the past.

2.1 Ramberg-Osgood non-dimensional formula

The designer would be at risk to use the stress strain curves that came with the purchased materials. Those curves supplied by venders were generously not of true values as a matter of advertisements. The purchased materials must be tested or validated in some way before usage. But each entire test is of time and cost, and there are numerous materials. Fortunately Ramberg-Osgood provided the non-dimensional fitting equation, that needs only as few as two test points to be able to construct the true stress strain curve from the vender supplied curve. It should be noted that the Ramberg-Osgood equation describes the non-linear relationship of the strain and stress at and near the yield point only, because this is where the designer is mostly concerned of when buildings, bridges become inelastically instable or fails. Using any formula one must know its limitation. No need to be smart to know that it is impossible to fit the whole complicated curve by only one exponential term. Later, the Ramberg-Osgood equation has been modified by MIL-HDBK or MMPDS to include \( f_0 \) (MMPDS-01, Chapter 9), or the correct notation \( f_{0.2ys} \) for the yield stress at 0.2 percent plastic strain. This modification is for better fit using a linear logarithm regression in the elastic-plastic transient of the stress strain curve, where the result from Ramberg-Osgood is sometimes not valid.

Today all stress strain curves published in the MIL-HBK or MMPDS were validated and are ready to be used. The Ramberg-Osgood parameters are also posted, but probably only for information, since the yield points have been determined and published. It is evident that this \( f_0 \) , or \( f_{0.2ys} \) , in the Ramberg-Osgood equation is nothing like the \( f_0 \) used by Cozonne in the procedure of calculating the plastic bending moment at different plastic state. As mentioned above, the whole effort of Ramberg-Osgood is only to accurately determine the yield strength, which is the most important property than all of a material, but nothing else.

The MMPDS also provides the complete stress strain curves. Since the exponential fit equation is valid only in the neighborhood of the yield point but never over the entire curve, these parameter are not seen
at those complete curves. It is seen very obvious from the exponential term, which ever increases with constantly positive derivative. Therefore the Ramberg-Osgood work has nothing related to plastic domain of the stress strain curve.

2.2 Cozzone’s plastic bending moment calculation

**Summary**

It is customary that designer uses the rectangular shape of the stress distribution for the full plastic material. It is a little bizarre, since it must be no strain at the neutral axis. However, as mentioned earlier, a representing full plastic stress profile across a section in Figure 1 does make sense. It shows basically zero strain at the neutral axis. Experimental investigations (no need to be cited) have supported the customarily used plastic shape factors and plastic hinges theory.

Integrating the stress area arm over the cross section area of a beam is the direct method of calculating the bending moment at a beam section. Cozzone simplified the procedure by providing a trapezoidal profile of stress distribution, whose small base \( f_0 \) (stress) is to be calculated and the large base has the same value \( f \) (stress) of the real curve, wherever the point on the curve one wishes to cut off. The value of \( f_0 \) is such that area of the trapezoid is the same as that under the specified section of the real curve, in other words, both area represent the same energy per unit volume. Whatever Cozzone did to come up to his trapezoidal stress profile, he had first to know the corresponding area under the real curve. This defines the equivalent linear stress at the corresponding state of material, as if a triangular stress profile per linear stress-moment equation is considered. A geometric shape factor is introduced to complete the Cozzone equation in the plastic state of material. Depending on where one wishes to the cut off the stress strain curve, at different state condition of the material, either in elastic state, partially plastic state or fully plastic state, it is evident the there are different \( f_0 \)’s corresponding to different states. Hence, in the elastic range \( f_0 \) is zero, the stress profile is simply triangular, and in the fully plastic range \( f_0 \) is close to \( F_{tu} \), the trapezoidal shape of the Cozzone stress profile is simply close to a rectangle having \( F_{tu} \) as the large base. Therefore, the maximum plastic bending moment per Cozzone is somewhat higher than the academic shape factor times yield bending moment.
Briefly saying, the work of Cozzone provides a quick way for calculating the equivalent stress at any elastic or plastic state with already provided $f_0$ stress. Then, knowing the cross section shape factor, the corresponding bending moment is evaluated. As a result, the Cozzone procedure gives the equivalent stress based on the area under the stress strain curve before the bending moment may be calculated.

**Example**

An example from Bruhn’s *Analysis and Design of Flight Vehicle Structures*, §C3.7 is repeated here for illustration. Same the 7075-T6 AL Ext material and the stress strain curve as in Figure 2 with the Cozzone’s $f_0$ curve which is the Fig. C3.17 in the same reference, partially repeated here in Figure 3.

![Figure 3](image-url)
Figure 3: 7075-T6 AL Ext (Bruhn’s Fig. C3.17 partially repeated)
From Bruhn's §C3.7, a rectangular bar of 0.25" wide and 1.5" deep. Calculate the yield moment $M_Y$. From the curve in Figure 3:

- $f_Y = 65$ ksi
- $f_0 = 29$ ksi
- shape factor $k = 1.5$ for a rectangle

then using equation

$F = f + f_0(k-1) = 79.5$ ksi

$M_Y = F \cdot I / c = 79.5 \cdot 0.0938 = 7.5$ in-kip
Suppose the designed ultimate strain not be exceeded \(0.03\) in/in strain at the extreme fiber. Again, from the curve in

\[
f_Y = 74.7 \text{ ksi} \quad f_0 = 61.2 \text{ ksi}
\]

then using equation

\[
F = f + f_0(k-1) = 105.3 \text{ ksi}
\]

\[
M_{\text{ULT}} = F\ell/c = 105.3 \times 0.0938 = 9.9 \text{ in-kip}
\]

Note that all resulting stresses are fictitious. They are simply the equivalent (or fictitious) linear stresses that produce the corresponding moments in the plastic state. These moments are however real.

### 2.3 Timoshenko text books

Many mechanical engineering text books written by Timoshenko were some time as bibles for students. The author usually left a complicated solution that he got stuck and said that the reader may follow the procedure the complete the rest. Why could he not?
In the *Strength of Materials*, Part II Chapter IX, §64, working on the a rectangular section, the author proved by formula that when the bending reaches the plastic ultimate bending, which is 1.5 times the yield bending moment, the curvature of the beam is infinite, or the radius of the curvature is zero. Is such thing in reality?

One should keep in mind that the math is only a tool used for the sciences. The math result is just a number, has no sense; its meaning is given by sciences. A solution of a matrix equation of an electronic network, that reduces to a some order polynomial, may yields a several results, real or complex, for a rheostat at one location, but only one may be accepted. Remember that a beam fails by the strain at the outer fiber, but not by the curvature.
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The factor 1.5 for the plastic bending of a rectangular section is based on the assumed rectangular profile of the stress distribution. Experimentally this factor is safe. Theoretically, the strain energy under the entire stress train curve is generally higher than 1.5 times the yield energy. To obtain the ultimate plastic bending, the factor is applied to the yield stress, not to the ultimate stress.

2.4 Plastic bending moment by piecewise integration

When Cozzone calculated the stress $f_0$ he must have integrated the area under the real curve, by some way, by using an area meter, by a theoretical fitting curve or by a numerical method whatever not known. Today, with the help of the computer, the task is quick, simple and easy.

One may use Microsoft XLS to fit the curve. However, it may run into a difficulty to fit an entire curve of multiple curvatures. Furthermore, it provides a polynomial only up to $6^{th}$ power, and has to use a lot of decimal places for accuracy. A slide fit may also be used. It fits not only point by point but also slope at each point. But it is not available from the Window XLS. One may write a best fit program using a polynomial of any order. The approach is to set up a rectangular matrix whose number of rows is equal to the number of points selected from the curve, and the number of columns is equal to the power number that one wishes, plus 1. The method is working with a pseudo-matrix, check for the generalized orthogonality of column vectors, minimizing the error norms, changing the power number to obtain a good correlation. So much is for that [1].

Briefly saying, a number of points of a curve to be fitted must be known before starting a fitting process, then once the fitting curve equation is found, an integration of this fitting curve equation is also to be performed. If it is so, then why not to perform a direct integration through the number of points that are already known. That is why a piecewise integration is the easy way. This method is as follows.

**Piecewise integration**

First, tabulate the stress strain curve point by point in ascending order. If one wishes to evaluate the bending moment up to strain $\varepsilon_b$ which corresponds to stress $f_b$, one should normalized the y coordinate on the basis of $\varepsilon_b$ to assure that the stresses and strains match along the y axis, as shown in Figure 4. Since the strain distribution is assumed linear, let only stress be non-linear, one may have

$$y = \frac{\varepsilon}{\varepsilon_b} c$$

(1)

where $c$ is the distance from the neutral axis to the outer fiber of the section.
Then the total bending moment is the sum of individual bending moments of differential trapezoidal areas, one shown by highlighted area in Figure 4, using the following formula:

\[ M = \sum \frac{y_{i+1} - y_i}{6} (2f_iy_i + 2f_{i+1}y_{i+1} + f_iy_{i+1} + f_{i+1}y_i) \left( \frac{w_{i+1} + w_i}{2} \right) \]  

(2)

The last parentheses account for the variation of the width, in the case of any size or shape of the cross section. In the case of a rectangle this term is constant. This presentation uses a unit half-height, c = 1, and unit width of 1.

![Figure 4: Stress profile across a beam section from the neutral axis to the outer fiber](image)

*Example*
Repeat the same the Bruhn’s problem in §C3.7 using Cozzone procedure. This problem has been previously summarized in §2.2 of this presentation. Rework the problem with Figure 3.

Detailed calculations may be referred to Table 1.

Applying the formula eq. (2) to $f_b = 65$ ksi and the corresponding $\varepsilon_b = 0.009$ in/in yields

$$\frac{1}{2} M \approx 26.2 c^2 \text{ in-kip}$$

This is referred to $c = 1$ and width $= 1$. For given $c = 0.75$” and $w = 0.25”$ the adjusted $\frac{1}{2} M$ is

$$\frac{1}{2} M \approx 26.2 * (0.75)^2 * (0.25) = 3.68 \text{ in-kip}$$

Remember that is a $\frac{1}{2}$ of the total moment, and assuming the section is symmetrical, so the moment is $M \approx 7.36 \text{ in-kip}$,

which is about 2% compared to 7.5 in-kip per Cozzone.

Remember that the approach is based on a rectangular section of dimensions of $w = 1$ unit wide by $2c$ deep. Thus, for the plastic “equivalent” linear stress calculation, one may have

$$F = \frac{6M}{(2c)^2} = \frac{6 * (2 * 26.2 * c^2)}{(2c)^2} = 78.6 \text{ ksi}$$
THE PLASTIC BENDING

which is about 1% compared to 79.5 ksi per Cozzone

Again, check the plastic bending at .03 in/in strain. See Table 1. This is referred to c = 1 and width = 1. For given c = .75" and w = .25" the adjusted ½ M is

\[ \frac{1}{2} M \approx 35.4 \times (0.75)^2 \times (0.25) = 4.98 \text{ in-kip} \]

Remember that is a ½ of the total moment, and assuming the section is symmetrical, so the moment is

\[ M \approx 9.96 \text{ in-kip} \]

which is less than 1% compared to 9.90 in-kip Cozzone calculated.

Remember that the approach is based on a rectangular section of dimensions of w = 1 unit wide by 2c deep. Thus, for the plastic “equivalent” linear stress calculation, one may have

\[ F = \frac{6M}{(2c)^2} = \frac{6 \times (2 \times 35.4 \times c^2)}{(2c)^2} = 106 \text{ ksi} \]

which is about 1% compared to 105 ksi per Cozzone

The Cozzone procedure gives first the stress, then from the stress the moment is evaluated. On the contrary, the integration method gives first the moment. The stress may be evaluated if needed. Again, all calculated stresses in the plastic domain are fictitious. The real stresses are only read from the stress strain curve. Only moments are real.

Two parameters have to be known in order to be able to use the Cozzone. These are the shape factor and the Cozzone \( f_0 \) stress. Not all materials have the \( f_0 \) information. Besides, the Cozzone procedure may apply only to symmetrical sections and symmetrical stress strain curves, only because of limited \( f_0 \) information. Hence, the Cozzone procedure has some limitation.
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Whatever method is used some accuracy is associated with it. In the numerical method the accuracy lies in how good the curve is read. However, if only a few points of the stress strain curve applied to the above piecewise integration yields a complete allowable bending moment at a beam cross section, then what the shape factor and the Cozzone $f_0$ stress are needed for?

Table 1 summarizes the calculation. An Excel spreadsheet is also provided.

Notes:

1. In this approach of integration the shape factor is not needed, because it is inherent in the process.

2. If the stress strain curve and the cross section are symmetrical, if the equation (2) is applied to a $\frac{1}{2}$ section, it would give $\frac{1}{2}$ of the total bending moment.

3. It should understand that this approach of integration gives the moment. If the input is the unit section area then the resulting moment must be converted to dimensions of the given area. If the input is based on the given area, then the result would be the net bending moment. From the resulting bending moment, the stress can be evaluated as “equivalent” linear stress. In the plastic domain all calculated stresses are fictitious, the only real stresses are those read from the stress strain curve. But, the moment is real.

4. Otherwise, in general for unsymmetrical section and non-linear stress strain curve, the equation (2) is also applied to the whole section to obtain a total moment. However, prior to applying equation (2), the neutral axis must be located such that the total force due to the bending stress distribution plus any axial force over the cross section must be zero.

Then place the origin of the stress strain curve (linear or non-linear) at the neutral axis.
Table 1: Plastic bending moment at yield strength and at .03 in/in strain

7075-T6 AL Ext

(Ref. Bruhn's *Analysis and Design of Flight Vehicle Structure* §C3.7)

AN EXAMPLE OF THE PLASTIC BENDING CALCULATION

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PER UNIT HEIGHT FROM NEUTRAL AXIS & UNIT WIDTH 35.43685185

Ref. Beam allowable bending moment.xls
3. CLOSURE

Applying 1.5 factor to the 65 ksi yield stress gives 97.5 ksi. This is the “equivalent” linear stress corresponding to customarily used for the ultimate plastic bending moment $M_{ULT}$. The “equivalent” linear stress 105 ksi calculated in the example presented in §2.2, or per Cozzone in Bruhn’s §C3.7, is 1.62 times higher than the yield stress, even this occurs at only .03 in/in strain. The material fails at more than 0.10 in/in strain. It proves that the customary plastic factor is still conservative. So what is the concern? However, it is not recommended to go higher than the customary plastic shape factors simply because of the instability in the plastic domain.

The piecewise integration method presented herein has following properties

1. It directly yields the basic bending moment of any cross section of a beam
2. There is no need of the shape factor and the Cozzone $f_0$
3. The method may be generalized and applied to any geometry and any kind of stress strain curve

Finally, in the design phase, one should avoid relying in the allowable plastic strength to size the part. It should save more room for the future rework or retrofit. Besides, a margin of safety related to the plastic domain should be clearly stated in percent of plasticity. It is of nonsense that a part has a negative margin from a linear static analysis, then just borrowing an additional plastic strength suddenly has a high margin.